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CORRIGENDUM TO OUR PAPER "THE ELLIPSOID METHOD AND ITS CONSEQUENCES IN COMBINATORIAL OPTIMIZATION"

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An error in the proof, and in the statement of a generalization, of the result that submodular setfunctions can be minimized over the subsets with odd cardinality is corrected.

In the paper mentioned in the title, we stated the following result. Let E be a finite set, $\mathcal{F}$ a collection of subsets of E closed under union and intersection (a lattice family) and f an integral valued submodular function defined on $\mathcal{F}$, i.e. a function $f\colon \mathcal{F}\to\mathbf{Z}$ such that $f(X)+f(Y)\geqq f(X\cap Y)+f(X\cup Y)$ holds for $X, Y\in\mathcal{F}$. Let $\mathcal{G}\subseteqq\mathcal{F}$ such that $E, \emptyset\notin\mathcal{G}$ and the following condition holds:

(∗) If $X\in\mathcal{G}$ and $Y\in\mathcal{F}-\mathcal{G}$ then either $X\cap Y\in\mathcal{G}$ or $X\cup Y\in\mathcal{G}$.

Then the minimum of f over the members of $\mathcal{G}$ can be found in polynomial time.

(It was assumed that one has a polynomial-time subroutine to compute $\bigcap\mathcal{F}$ and $\bigcup\mathcal{F}$, another one to decide if $\mathcal{F}$ contains a set containing a given $x\in E$ but missing another given $y\in E$, yet another polynomial time subroutine to check if $Y\in\mathcal{G}$ and one more polynomial-time subroutine to compute $f(X)$ for $X\in\mathcal{F}$. Further, it was also assumed that a positive upper bound B on $|f(X)|$ is given a priori; Fujishige and Tomizawa observed that such a bound can be computed in polynomial time.)

The most important special case of this result is that the minimum of a submodular function over a lattice family of subsets can be found in polynomial time (i.e. $\mathcal{F}=\mathcal{G}$). This special case was proved first separately. Another special case of interest is when $\mathcal{G}$ is the collection of odd cardinality members of $\mathcal{F}$. This generalizes the result of Padberg and Rao [3] which finds a minimum weight odd cut in a graph.

András Frank pointed out to us that there is a gap in the proof of the general theorem. It turns out, in fact, that it is not true as stated. In this note we first give the counterexample, then proceed to formulate the slightly stronger hypotheses on $\mathcal{G}$ under which a similar theorem can be proved. All special cases of interest mentioned in the original paper remain valid; however, the proof (i.e. the algorithm) turns out considerably more involved.

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Example 1. Let E be a finite set, $\mathcal{F}=2^E$, f the rank function of a matroid on E of rank k, and $\mathcal{G}=\{X\subseteq E\colon |X|\geqq k\}$. Then $\mathcal{F}$ and $\mathcal{G}$ satisfy the conditions in the "theorem" mentioned above. Moreover, we have

$$\min\{f(X)\colon X\in\mathcal{G}\}\ \begin{matrix} = k, & \text{if the matroid is } k\text{-uniform}, \\ < k, & \text{else.}\end{matrix}$$

Now it is easy to see by an "oracle" argument (see Jensen and Korte [2]), that it takes exponential time in the worst case to decide if a given matroid is uniform. So it takes exponential time to find the minimum of f over $\mathcal{G}$.

This construction is not a counterexample to the "theorem" stated above, since $E\in\mathcal{G}$. But we can repair this by adding a new element e and letting $E'=E\cup\{e\}$, $f(E')=f(E)$, $\mathcal{F}'=\mathcal{F}$ and $\mathcal{G}'=\mathcal{G}$.

Consider the following condition on $\mathcal{F}$ and $\mathcal{G}$.

$(**)$ If three of the sets $X, Y, X\cup Y$ and $X\cap Y$ belong to $\mathcal{F}-\mathcal{G}$ then the fourth also belongs to $\mathcal{F}-\mathcal{G}$.

The following two propositions show the relationship between conditions $(*)$ and $(**)$.

Proposition 2. $(**)$ *implies* $(*)$.

Proof. Suppose that $X\in\mathcal{G}$ and $Y\in\mathcal{F}-\mathcal{G}$. Then since $\mathcal{F}$ is a lattice family, i.e. closed under union and intersection, we have that $X\cup Y$, $X\cap Y\in\mathcal{F}$. If both $X\cup Y$, $X\cap Y\in\mathcal{F}-\mathcal{G}$ then exactly three of the sets $X, Y, X\cup Y$ and $X\cap Y$ belong to $\mathcal{F}-\mathcal{G}$, which contradicts $(**)$. So at least one of $X\cup Y$ and $X\cap Y$ must belong to $\mathcal{G}$. ▮

Proposition 3. *If* $\mathcal{F}=2^E$ *and* $\emptyset, E\notin\mathcal{G}$ *then* $(*)$ *implies* $(**)$.

Proof. Suppose that three of the sets $X, Y, X\cup Y$ and $X\cap Y$ belong to $\mathcal{F}-\mathcal{G}$; we show that so does the fourth. There are essentially two cases to consider.

Case 1. $X\cup Y, X\cap Y$ and X belong to $\mathcal{F}-\mathcal{G}$. Then $Y\in\mathcal{G}$ would immediately contradict $(*)$.

Case 2. X, Y and (say) $X\cap Y$ belong to $\mathcal{F}-\mathcal{G}$. Consider $Y'=Y\cup(E-X)$. Since $X\in\mathcal{F}-\mathcal{G}$, $X\cap Y'=X\cap Y\in\mathcal{F}-\mathcal{G}$ and $X\cup Y'=E\in\mathcal{F}-\mathcal{G}$, we have by $(*)$ that $Y'\in\mathcal{F}-\mathcal{G}$. But also $Y'\cap(X\cup Y)=Y\in\mathcal{F}-\mathcal{G}$ and $Y'\cup(X\cup Y)=E\in\mathcal{F}-\mathcal{G}$, and hence again by $(*)$, $X\cup Y\in\mathcal{F}-\mathcal{G}$. ▮

The main theorem in the Corrigendum is the following. Note that we had to strengthen the hypothesis by replacing $(*)$ by $(**)$; but we have also weakened it by dropping the hypothesis that $\emptyset, E\notin\mathcal{G}$.

Theorem 4. *Let* $\mathcal{F}$ *and* $\mathcal{G}$ *be two families of subsets of a finite set E, such that* $\mathcal{F}$ *is closed under union and intersection,* $\mathcal{G}\subseteq\mathcal{F}$ *and* $(**)$ *is fulfilled. Let f be a submodular function on* $\mathcal{F}$. *Then one can find the minimum of f over* $\mathcal{G}$ *in polynomial time.*

Corollary 5. *Let $\mathcal{F}$ be a family of subsets of a finite set E closed under union and intersection, and let $a, b \in \mathbf{Z}$. Let f be a submodular function on $\mathcal{F}$. Then* $\min \{f(X)\colon X \in \mathcal{F},\ |X| \not\equiv a \pmod b\}$ *can be found in polynomial time.* ∎

Corollary 6. *Let $\mathcal{F}$ be a family of subsets of a finite set E closed under union and intersection, and let $\mathcal{A}$ be an antichain in $\mathcal{F}$. Let f be a submodular function on $\mathcal{F}$. Then one can find the minimum of f over $\mathcal{F}-\mathcal{A}$ in polynomial time.* ∎

Proof. *I.* First we remark that one may assume that $\mathcal{F}=2^E$. In fact, we may assume that $E \in \mathcal{F}$, since those elements of E not contained in the unique largest member of $\mathcal{F}$ (and hence in any member of $\mathcal{F}$) can be deleted. Let, for each $X \subseteq E$, $I(X)$ denote the unique smallest member of $\mathcal{F}$ containing X. Define a set-function g on 2^E by $g(X)=f(I(X))+2B|I(X)-X|$. Then g is submodular, and $g(X)=f(X)$ for every $X \in \mathcal{F}$. Furthermore, define $\mathcal{G}_0=\mathcal{G} \cup (2^E-\mathcal{F})$. Then, obviously, $(**)$ is fulfilled with 2^E in place of $\mathcal{F}$ and $\mathcal{G}_0$ in place of $\mathcal{G}$. By the definition of B, $g(X)>f(Y)$ for any $X \subseteq E$, and $Y \in \mathcal{F}$, unless $X=I(X)$, i.e. $X \in \mathcal{F}$. Hence

$$\min \{g(X)\colon X \in \mathcal{G}_0\} = \min \{f(X)\colon X \in \mathcal{G}\}.$$

Thus we shall assume in the sequel that $\mathcal{F}=2^E$.

II. Let $T(\mathcal{G})=T=\{x \in E\colon \{x\} \in \mathcal{G}\}$, $S=\{x \in E\colon E-x \in \mathcal{G}\}$, $t=|T|$. Let us make a few observations about these sets.

Claim 1. If $\emptyset \notin \mathcal{G}$ and $E \notin \mathcal{G}$, then $S=T$.
In fact, if e.g. $\{x\} \in \mathcal{G}$ but $E-x \notin \mathcal{G}$ then by $(*)$, one of $\emptyset$ and E must belong to $\mathcal{G}$.

Claim 2. If $\emptyset \notin \mathcal{G}$, then for any $A \subseteq E$, one has $A \in \mathcal{G}$ iff $A \cap T \in \mathcal{G}$.
For, let $A \cap T \in \mathcal{G}$, and choose a maximal set A' such that $A \cap T \subseteq A' \subseteq A$ and $A' \in \mathcal{G}$. If $A \neq A'$ then choose any $u \in A-A'$. Then $A' \in \mathcal{G}$, $\{u\} \notin \mathcal{G}$ and $A' \cap \{u\}=\emptyset \notin \mathcal{G}$. Hence by $(**)$, $A' \cup \{u\} \in \mathcal{G}$, which contradicts the maximality of A'. So $A'=A$ and thus $A \in \mathcal{G}$. The reverse implication follows by the same argument. ∎

III. We now describe the algorithm in the case when $\emptyset \notin \mathcal{G}$ and $E \notin \mathcal{G}$. This will be the most difficult case, in the other cases the necessary modifications will simplify the argument.

So let $E, \emptyset \notin \mathcal{G}$. We shall describe an algorithm to minimize f over $\mathcal{G}$ in time $O(|E|^3 p(|E|, \log_2 B))$ where $p(n, \log_2 B)$ is an upper bound on the time needed to minimize a submodular setfunction over all subsets of a set of cardinality at most n, whose values are bounded by B. If $I=\emptyset$ then $\mathcal{G}=\emptyset$, and we are done.

First we find a set A such that $T \cap A \neq \emptyset$, $T-A \neq \emptyset$ and $f(A)$ is minimal. This can be done by $t(t-1)$ applications of the submodular function minimization algorithm, by applying it to find the minimum of f over all sets A with $\{u\} \subseteq A \subseteq E-v$ for all pairs $u, v \in T$, $u \neq v$ (note that $|T| \geqq 2$ by Claim 2). We call A a *splitter*.

If $A \in \mathcal{G}$ then we are done. In fact, any set $A' \in \mathcal{G}$ satisfies $T \cap A' \neq \emptyset$ and $T-A' \neq \emptyset$, by Claims 1 and 2 above, and so $f(A) \leqq f(A')$ for each $A' \in \mathcal{G}$, by the choice of A. So we may assume that $A \notin \mathcal{G}$.

Let $A_1=A \cap T$, $A_2=T-A$. It follows by Claim 2 that $A_i \notin \mathcal{G}$. Take two new elements a_1 and a_2, and define $E_i=(E-A_i) \cup \{a_i\}$,

$$f_i(X)=\begin{cases} f(X) & \text{if } X \subseteq E-A_i, \\ f(X-a_i \cup A_i) & \text{if } X \subseteq E_i,\ a_i \in X, \end{cases}$$

and

$$\mathcal{G}_i = \{X\in\mathcal{G}\colon X \subseteq E-A_i\}\cup\{X \subseteq E_i\colon a_i \in X,\ X-a_i\cup A_i\in\mathcal{G}\}.$$

It is straightforward to check that f_i is a submodular setfunction on the subsets of E_i and $\mathcal{G}_i \subseteq 2^{E_i}$ satisfies $(**)$. Furthermore, $\emptyset,\ E_i \notin \mathcal{G}_i$.

We claim that

$$\min\{f(X)\colon X\in\mathcal{G}\} = \min\{\min\{f_i(X)\colon X\in\mathcal{G}_i\}\colon i = 1, 2\}.$$

The sign $\leqq$ is obvious: if X is a set minimizing the right hand side, and say $X\in\mathcal{G}_1$, then either $X'=X$ or $X'=X-a_1\cup A_1$ is a set such that $X'\in\mathcal{G}$ and $f(X')=f_1(X)$.

To show the reverse inequality, let $X\in\mathcal{G}$ be a set minimizing the left hand side. If $X\cap A_1=\emptyset$ then $X\in\mathcal{G}_1$ and $f(X)=f_1(X)$, and so we are done. So we may assume that $X\cap A_1\neq\emptyset$. Similarly, if $A_2\subseteq X$ then $X'=X-A_2\cup\{a_2\}\in\mathcal{G}_2$ and $f(X)=f_2(X')$, and the assertion follows again. So we may also assume that $A_2\not\subseteq X$.

Since $X\in\mathcal{G}$ but $A\notin\mathcal{G}$, it follows by $(*)$ that either $X\cup A\in\mathcal{G}$ or $X\cap A\in\mathcal{G}$. We treat the first case; the second is similar.

Since $X\cup A\in\mathcal{G}$, we have that $f(X\cup A)\geqq f(X)$ by the choice of X. Furthermore, since $(X\cap A)\cap T=X\cap A_1\neq\emptyset$, and $T-(X\cap A)\supseteq T-A\neq\emptyset$, we have that $f(X\cap A)\geqq f(A)$ by the choice of A. Using the submodularity of f we get $f(X\cup A)+f(X\cap A)\leqq f(X)+f(A)$. So equality must hold everywhere, in particular $f(X\cup A)=f(X)$. So $X\cup A$ is also minimizing f over $\mathcal{G}$. But then $X'=X\cup A-A_1\cup\{a_1\}\in\mathcal{G}_1$ and $f_1(X')=f(X)$, whence the assertion follows again.

Thus we have found that in order to find the minimum of f over $\mathcal{G}$, it suffices to find the minimum of f_1 over $\mathcal{G}_1$ and the minimum of f_2 over $\mathcal{G}_2$. Going on similarly, we can split each of these subproblems into two, or find the minimizing set right away. This describes the algorithm to find the minimum of f over $\mathcal{G}$.

We still have to show that this algorithm runs in polynomial time. The main observation is that $T(\mathcal{G}_i)=A_i$, i.e. the set of singletons in $\mathcal{G}$ is split into two non-empty parts to obtain the sets of singletons in $\mathcal{G}_1$ and $\mathcal{G}_2$. This implies that the number of splitters to compute during the procedure is at most $t-1<|E|$. To compute one splitter takes less that $|E|^2$ submodular function minimizations, and so the whole algorithm takes only about $|E|^3p(|E|, \log B)$ time. This completes the case when $\emptyset, E\notin\mathcal{G}$.

IV. Suppose that $\emptyset\notin\mathcal{G}$ but $E\in\mathcal{G}$. We follow the same argument as in III with slight modifications. First we find a set A such that $T\cap A\neq\emptyset$ and $f(A)$ is minimal. This is easily achieved by t applications of the submodular function minimization algorithm.

If $A\in\mathcal{G}$ then we are done again, so suppose that $A\notin\mathcal{G}$. Set $A_1=A\cap T$; take a new element a_1 and define

$$E_1 = (E-A_1)\cup\{a_1\}$$

$$f_1(X) = \begin{cases} f(X) & \text{if } X\subseteq E-A_1, \\ f(X-a_1\cup A_1) & \text{if } X\subseteq E_1,\ a_1\in X, \end{cases}$$

and

$$\mathcal{G}_1 = \{X\in\mathcal{G}\colon X \subseteq E-A_1\}\cup\{X\subseteq E_1\colon a_1\in X,\ X-a_1\cup A_1\in\mathcal{G}\}.$$

Also define $E_2=A$, $f_2(X)=f(X)$ for $X\subseteq A$ and $\mathcal{G}_2=2^A\cap\mathcal{G}$. Then f_i is submodular on the subsets of E_i and $\mathcal{G}_i\subseteq 2^{E_i}$ satisfies $(**)$. It also follows just like in part III that

$$\min\{f(X)\colon X\in\mathcal{G}\} = \min\bigl\{\min\{f_i(X)\colon X\in\mathcal{G}_i\}\colon i=1,2\bigr\}.$$

So again it suffices to minimize f_1 over $\mathcal{G}_1$ and f_2 over $\mathcal{G}_2$. The second of these tasks can be solved in polynomial time by III. Since $|T(\mathcal{G}_1)|=|A_1|<|T|$, we are finished by induction.

V. The cases when $\emptyset\in\mathcal{G}$ but $E\notin\mathcal{G}$ and when $\emptyset$, $E\in\mathcal{G}$ can be treated similarly. ∎

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